f–HARMONIC MAPS AND APPLICATIONS TO GRADIENT RICCI SOLITONS

MICHELE RIMOLDI AND GIONA VERONELLI

ABSTRACT. In this paper we study f–harmonic maps from non–compact manifolds into non–positively curved ones. Notably, we prove existence and vanishing results which generalize to the weighted setting part of Schoen and Yau's theory of harmonic maps. As an application, we deduce information on the topology of manifolds with lower bounded ∞–Bakry–Émery Ricci tensor, and in particular of steady and expanding gradient Ricci solitons.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Let $(M^m, \langle\,,\,\rangle_M)$ and $(N^n, \langle\,,\,\rangle_N)$ be complete Riemannian manifolds, $\dim M = m \geq 2$, $\dim N = n$. Let $f : M \to \mathbb{R}$ be a smooth function. A map $u : M \to N$ is said to be (weakly) f–harmonic if $u|_\Omega$ is a critical point of the f–energy

$$E_f(u) = \frac{1}{2} \int_\Omega e^{-f} |du|^2_{HS} dV_M$$

for every compact domain $\Omega \Subset M$. Here $|\cdot|_{HS}$ denotes the Hilbert–Schmidt norm on the set $T^*M \otimes u^{-1}TN$ of the vector–valued 1–forms along the map u and dV_M stands for the canonical Riemannian volume form on M. When u is C^2–regular, the Euler–Lagrange equation for the energy functional E_f is the f–harmonic maps equation [15, 6]

$$\tau_f u := e^f \operatorname{div}(e^{-f} du) = \tau u - i_{\nabla f} du = 0,$$

where $\tau u = \operatorname{div} du$ is the standard tension field of u, so that $\tau_f u$ is naturally named f–tension field of u. Here i denotes the interior product on 1–forms, i.e. $i_{\nabla f} du = du(\nabla f)$, while $-\operatorname{div}$ stands for the formal adjoint of the exterior differential d with

respect to the standard L^2 inner product on vector–valued 1–forms. The study of f–harmonic maps began with A. Lichnerowicz in 1969 [15] and J. Eells and L. Lemaire in 1977, [8], but apparently this subject has been very poorly investigated later. Let us just recall the recent works of N. Course [6, 7], especially about f–harmonic flow on surfaces, and Y.–L. Ou [21] about f–harmonic morphisms. A possible reason for such a lack of interest could be that the f–harmonicity of a map u defined on a Riemannian manifold M is equivalent to the harmonicity of u on some related manifolds, see Proposition 2.4 and Proposition 2.5 below. For instance, if $m \geq 3$ then $u : (M^m, \langle\,,\,\rangle_M) \to N$ is f–harmonic if and only if $u : \left(M^m, e^{-\frac{2f}{m-2}} \langle\,,\,\rangle_M\right) \to$ N is harmonic. Accordingly f–harmonic maps could appear not so interesting as geometric objects in their own. Nevertheless, the interaction of f–harmonicity with curvature conditions looks promisingly and justifies their study in order to deduce information on weighted manifolds and gradient Ricci solitons. For this reason, notation used in this paper is that of weighted manifolds, e.g. [33, 32] and not the one introduced so far for f–harmonic maps, where often e^{-f} is replaced by f [15, 6, 21].

A weighted manifold is a Riemannian manifold $(M^m, \langle\,,\,\rangle_M)$ endowed with a weighted volume form $e^{-f}dV_M$, for some smooth function $f : M \to \mathbb{R}$. Associated to a weighted manifold $(M^m, \langle\,,\,\rangle_M, e^{-f}dV_M)$ there is also a natural divergence form second order diffusion operator: the f–Laplacian. This is defined on $u \in C^2(M)$ by $\Delta_f u = e^f \operatorname{div}(e^{-f}\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle_M$ and we can note that, for real–valued functions, $\Delta_f u = \tau_f u$. A natural question that arises in the setting of weighted manifolds is what is the right concept of curvature on these spaces. Actually there is not a canonical choice. Good choices are those that reveal interplays with metric and topological properties of the space, see e.g. [17, 33, 32]. We are interested in the ∞–Bakry–Émery Ricci tensor ${}^M\operatorname{Ric}_f = {}^M\operatorname{Ric} + \operatorname{Hess} f$, which was first introduced by A. Lichnerowicz in [16] and later by D. Bakry and M. Émery in [1]. Recently it has been found that this curvature tensor is strictly related with geometric objects whose importance is outstanding in mathemathics. Imposing the constancy of ${}^M\operatorname{Ric}_f$, one introduces on the manifold an additional structure which goes under the name of gradient Ricci soliton structure. Namely, recall that, given a Riemannian manifold $(M, \langle\,,\,\rangle)$, a Ricci soliton structure on M is the choice of a smooth vector field X (if any) satisfying the soliton equation $\operatorname{Ric} + \frac{1}{2}\mathcal{L}_X \langle\,,\,\rangle = \lambda \langle\,,\,\rangle$, for some $\lambda \in \mathbb{R}$. The Ricci soliton $(M, \langle\,,\,\rangle, X)$ is said to be shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. In the special case where $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we have that $\operatorname{Ric}_f = \lambda \langle\,,\,\rangle$, and we say that $(M, \langle\,,\,\rangle, \nabla f)$ is a gradient Ricci soliton with potential f. The importance of gradient Ricci solitons is due to Perelman's solution of Poincaré conjecture. They correspond to "self–similar" solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the flow.

Mimicking the theory developed by J. Eells and J. H. Sampson [10] and P. Hartman [14], in the seminal paper [15] Lichnerowicz use f–harmonic maps to deduce topological information on compact $(m \geq 3)$–dimensional weighted manifolds satisfying ${}^M\operatorname{Ric}_f \geq 0$. In this paper, following the theory for harmonic maps developed by R. Schoen and S.–T. Yau [28], we extend the approach of Lichnerowicz in order to study the topology of complete manifolds of dimension $m \geq 2$ with non–negative Bakry–Emery Ricci tensor ${}^M\operatorname{Ric}_f \geq 0$ and of expanding gradient Ricci solitons with

a suitable control on the scalar curvature. This problem is particularly interesting since very poor information on the topology of this class of manifolds is known, see Section 5.

The main result we obtain is the following.

Theorem A. *Let M be a complete non–compact Riemannian manifold and N a compact Riemannian manifold with $^N \mathrm{Sect} \leq 0$. Let $f \in C^\infty(M)$ and suppose $^M \mathrm{Ric}_f \geq -k^2(x)$ for some $0 \leq k \in C^\infty(M)$. Consider a continuous map $u : M \to N$ with finite f–energy $E_f(u) < +\infty$. Then u is homotopic to a constant provided*

$$\lambda_1(-\Delta_f - Hk^2) \geq 0 \tag{1}$$

for some $H > 1$ and at least one of the following assumption is satisfied

    (a) *k does not vanish identically;*
    (b) *there exists a constant $C > 0$ such that $|f| \leq C$;*
    (c) *f is convex and the set of its critical points is unbounded;*
    (d) *$\mathrm{Vol}_f(M) := \int_M e^{-f} dV_M = +\infty$;*
    (e) *there is a point $q_0 \in M$ such that $^M \mathrm{Ric}_f |_{q_0} > 0$;*
    (f) *there is a point $q_1 \in M$ such that $^M \mathrm{Ric}(X, X)|_{q_1} \neq 0$ for all $0 \neq X \in T_{q_1} M$.*

Moreover if

    (g) *$^N \mathrm{Sect} < 0$,*

we can conclude that u is homotopic either to a constant or to a totally geodesic map whose image is contained in a geodesic of N.

Remark 1.1. By Rayleigh characterization, the spectral assumption (1) is equivalent to ask that

$$\int_M |\nabla \varphi|^2 e^{-f} dV_M - \int_M Hk^2 \varphi^2 e^{-f} dV_M \geq 0 \tag{2}$$

for all $\varphi \in C_c^\infty(M)$. Then, in case $^M \mathrm{Ric}_f \geq 0$, the assumption (1) is trivially satisfied, and the theorem holds true under conditions (b) to (g).
The non–weighted $k \not\equiv 0$ version of Theorem A is due to [23]. See also [27], where the lower bound on H is improved, and Remark 4.2 below for a comment on the lower bound for H.

As pointed out by Schoen and Yau in the non–weighted case [28], Theorem A has the following topological implications.

Corollary B. *Let M^m be a complete non–compact m–dimensional Riemannian manifold satisfying $^M \mathrm{Ric}_f \geq -k^2$ for some function k satisfying (1). Let $D \subset M$ be a compact domain in M with smooth, simply connected boundary. Then,*

    (i) *if one of the condition (a) to (f) of Theorem A is satisfied, then there is no non–trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non–positive sectional curvature;*
    (ii) *each homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold N with strictly negative sectional curvature $^N \mathrm{Sect} < 0$ is either trivial or maps all $\pi_1(D)$ into a cyclic subgroup of $\pi_1(N)$.*

Remark 1.2. Note that Corollary B (ii) and Corollary B (i) in the assumptions (a), (e) or (f) hold not asking f to be neither bounded nor convex. This is a reason of interest in the approach we propose, since it permits to deal with cases for which the techniques introduced so far seem to be unapplicable. See Theorem 5.1 below.

The $k \equiv 0$ case of Corollary B directly applies to study the topology of complete steady gradient Ricci solitons, where few information is known, as discussed in Section 5 below. On the other hand, the flexibility given by the spectral assumption permits to deduce information also concerning expanding Ricci solitons. In particular, according to an idea in [18], there exist situations in which the spectral assumption is implied by more geometrical curvature condition.

Theorem C. *Let $(M^m, \langle\,,\,\rangle_M, \nabla f)$ be a complete non–trivial expanding gradient Ricci soliton with scalar curvature $^M S > (m-1)\lambda$. Let $D \subset M$ be a compact domain in M with smooth, simply connected boundary. Then, there is no non–trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non–positive sectional curvature.*

The paper is organized as follows. In Section 2 we discuss the relations between the weighted and the non–weighted setting, and in particular under which conditions f–harmonic maps on a manifold M can be interpreted as harmonic maps on some related manifold. This permits to deduce the existence of a smooth f–harmonic representative in the homotopy class of a given finite f–energy map, provided N is compact and $^N \text{Sect} \leq 0$. Also, some information about the uniqueness of the representative is given. In Section 3 we prove a vanishing result for finite f–energy f–harmonic maps, in the special easier case $k \equiv 0$. Then, in Section 4 we discuss the spectral assumption (1) and provide the changes needed to complete the proof of Theorem A. Finally, in Section 5 we deduce geometrical applications to weighted manifolds and gradient Ricci solitons.

2. Existence and relations between the weighted and the non–weighted setting

This section aims to give the proof of the following existence result. To obtain this, we will formalize some useful links between harmonic and f–harmonic maps.

Theorem 2.1. *Let M^m and N^n be Riemannian manifolds, $m \geq 2$. Assume N is compact and $^N \text{Sect} \leq 0$. Then any homotopy class of maps from M into N containing a continuous map of finite f–energy contains a smooth f–harmonic map minimizing the f–energy in the homotopy class.*

Remark 2.2. When M is compact and $m \geq 3$, Theorem 2.1 is due to [8], p.48.

Remark 2.3. Instead of $^N \text{Sect} \leq 0$ it would be enough for N to be a $K(\pi, 1)$–space and to admit no non–trivial minimizing tangent maps or r–spheres for $2 \leq r \leq m - 1$. For instance this is the case if the universal cover of N supports a strictly convex exhaustion function; see also [4, 34].

In dimension $m > 2$ the proof is pretty easy. First, by straightforward computations, [15], one can prove that

Proposition 2.4. *A map $u : (M^m, \langle\,,\,\rangle_M) \to (N^n, \langle\,,\,\rangle_N)$, $m \geq 3$, is f–harmonic if and only if $u : (M^m, e^{-\frac{2f}{m-2}} \langle\,,\,\rangle_M) \to (N^n, \langle\,,\,\rangle_N)$ is a harmonic map.*

In [4], F. Burstall proved that if $^N \text{Sect} \leq 0$, then in the homotopy class of each finite energy map there exists a smooth harmonic representative which minimizes the energy in the homotopy class. As a matter of fact, even if the result is there stated for complete manifolds, the proof does not require the underlying manifold

M to be complete. Accordingly, from Proposition 2.4 and Theorem 5.2 in [4] we get the validity of Theorem 2.1 when $m > 2$. On 2-dimensional manifolds, it is easily seen that (f-)energy is conformally invariant so that this approach does not work. Then, we follow a different strategy which permits to obtain the result in all dimension $m \geq 2$. Namely, it was suggested in Section 1.2 of [6] that f–harmonicity on M is expected to correspond to harmonicity on some higher dimensional warped product manifold. We will make this fact explicit.

We consider the warped product $\bar{M}^n = M \times_h \mathbb{T}$, where $h := e^{-f}$ and $\mathbb{T} = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, so that $\mathrm{Vol}(\mathbb{T}) = 1$. Here and on, each point in $\bar{M}$ is individuated by its projections x on M and t on $\mathbb{T}$. Moreover we recall that the metric on $\bar{M}$ is given by $\langle \, , \, \rangle_{\bar{M}}(x,t) = \langle \, , \, \rangle_M(x) + h^2(x)dt^2$. Throughout the following proofs, $\{E_i\}_{i=1}^n$ is a local orthonormal frame at $(x,t) \in \bar{M}$ such that $\{E_j\}_{j=1}^{n-1}$ is a local orthonormal frame at $x \in M$ and $E_n = h^{-1}\frac{\partial}{\partial t} \in T_t\mathbb{T}$.

Even if this is not necessarily used in the proof of Theorem 2.1 given below, we start pointing out the explicit relation, of its own interest, between the f–tension field on M and the tension field on $\bar{M}$.

Proposition 2.5. *Given a C^2 map $v : M \to N$, define the C^2 map $\bar{v} : \bar{M} \to N$ as $\bar{v}(x,t) := v(x)$ for all $(x,t) \in \bar{M}$. Then*

$$\tau\bar{v}(x,t) = \tau_f v(x).$$

In particular, v is f–harmonic if and only if $\bar{v}$ is harmonic.

Proof. Following the rules of covariant derivatives on warped products, see [20] p. 206, we can compute

$$\left({}^{\bar{M}}\nabla_{E_j}E_j \right)(x,t) = \left(\left({}^M\nabla_{E_j}E_j \right)(x), 0 \right)$$

for $j = 1, \ldots, n-1$, and

$$\left({}^{\bar{M}}\nabla_{E_n}E_n \right)(x,t) = \left(-\frac{{}^M\nabla h(x)}{h}, \left({}^{\mathbb{T}}\nabla_{E_n}E_n \right)(t) \right) = \left({}^M\nabla f(x), 0 \right).$$

Moreover, by definition of $\bar{v}$ we have

$$\left({}^N\nabla_{d\bar{v}(E_j)}d\bar{v}(E_j) \right)(x,t) = \left({}^N\nabla_{dv(E_j)}dv(E_j) \right)(x)$$

for $j = 1, \ldots, n-1$, and

$$\left({}^N\nabla_{d\bar{v}(E_n)}d\bar{v}(E_n) \right)(x,t) = 0.$$

Then, we get

$$\mathrm{Hess}\,\bar{v}|_{(x,t)}(E_j, E_j) = \left({}^N\nabla_{d\bar{v}(E_j)}d\bar{v}(E_j) - d\bar{v}({}^{\bar{M}}\nabla_{E_j}E_j) \right)(x,t)$$
$$= \left({}^N\nabla_{dv(E_j)}dv(E_j) - dv({}^M\nabla_{E_j}E_j) \right)(x)$$

for $j = 1, \ldots, n-1$, and

$$\mathrm{Hess}\,\bar{v}|_{(x,t)}(E_n, E_n) = \left({}^N\nabla_{d\bar{v}(E_n)}d\bar{v}(E_n) - d\bar{v}({}^{\bar{M}}\nabla_{E_n}E_n) \right)(x,t)$$
$$= -dv\left(({}^M\nabla f)(x) \right).$$

We thus obtain

$$
\begin{aligned}
\tau \bar{v}(x,t) &= \sum_{i=1}^{n} \operatorname{Hess} \bar{v}|_{(x,t)}(E_i, E_i) \\
&= \sum_{j=1}^{n-1} \operatorname{Hess} v|_x(E_j, E_j) - dv\left(({}^{M}\nabla f)(x)\right) \\
&= \tau v(x) - dv\left(({}^{M}\nabla f)(x)\right) = \tau_f v(x).
\end{aligned}
$$

$\square$

We have shown that any f–harmonic map on M is harmonic when trivially extended to $\bar{M} = M \times_{e^{-f}} \mathbb{T}^1$. Clearly, the converse does not hold, since in general a harmonic map from $\bar{M}$ to N does depend on t. To this end, it suffices to consider the example given by $M = \mathbb{R}$, $f \equiv 0$ and $N = \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, and it is easily seen that the locally isometric covering maps $P : \mathbb{R} \times \mathbb{T} = \bar{M} \to N = \mathbb{T}^2$ is harmonic, but it has not the form $P(x,t) \equiv P(x,T)$ for any $T \in \mathbb{T}$. Nevertheless, as the following proof shows, this converse property holds true for some specific harmonic maps on $\bar{M}$ which minimize energy in their homotopy class.

Proof (of Theorem 2.1). **Step a.** Let $v : M \to N$ be a continuous map satisfying $E_f(v) < +\infty$ and define the continuous function $\bar{v} : \bar{M} \to N$ as $\bar{v}(x,t) := v(x)$. We note that

$$
\begin{aligned}
|d\bar{v}|^2_{HS(\bar{M},N)} &= \sum_{i=1}^{n} \langle d\bar{v}(E_i), d\bar{v}(E_i) \rangle_N \\
&= \sum_{j=1}^{n-1} \langle dv(E_j), dv(E_j) \rangle_N = |dv|^2_{HS(M,N)},
\end{aligned}
$$

and the latter relation holds in the weak sense for $v, \bar{v} \in C^0$. Since v has finite f–energy and $\operatorname{Vol}(\mathbb{T}) = 1$, we can apply Fubini's theorem to get

$$
\begin{aligned}
(3) \qquad E^{\bar{M}}(\bar{v}) &= \int_{\bar{M}} |d\bar{v}|^2_{HS(\bar{M},N)} dV_{\bar{M}} = \int_M e^{-f(x)} \int_{\mathbb{T}} |dv|^2_{HS(M,N)} dt dV_M \\
&= \int_M e^{-f(x)} |dv|^2_{HS(M,N)} dV_M = E_f^M(v)
\end{aligned}
$$

So $\bar{v}$ is a continuous finite energy map from $\bar{M}$ to a compact manifold N with $^N \operatorname{Sect} \leq 0$, and according to [4] we know that there exists a smooth harmonic map $\bar{u} : \bar{M} \to N$ which minimizes the energy in its homotopy class.
Step b. In this step, we prove that we can choose $\bar{u}$ such that it has the form $\bar{u}(x,t) = u(x)$ for some smooth map $u : M \to N$. In fact, since $E^{\bar{M}}(\bar{u}) < +\infty$, we can apply Fubini's theorem to get

$$
\begin{aligned}
\infty > E^{\bar{M}}(\bar{u}) &= \int_M e^{-f(x)} \int_{\mathbb{T}} |d\bar{u}|^2_{HS(\bar{M},N)}(x,t) dt dV_M(x) \\
&= \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\bar{u}|^2_{HS(\bar{M},N)}(x,t) dV_M(x) \right) dt,
\end{aligned}
$$

and we can choose $T \in \mathbb{T}$ such that

$$(4) \qquad \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\bar{u}|_{HS(\bar{M},N)}^2 (x,T) dV_M(x) \right) dt$$

$$\leq \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\bar{u}|_{HS(\bar{M},N)}^2 (x,t) dV_M(x) \right) dt.$$

We can define a further smooth map $\tilde{u} : \bar{M} \to N$ as $\tilde{u}(x,t) := \bar{u}(x,T)$ for all $(x,t) \in \bar{M}$. Since $d\tilde{u}(E_n) = 0$, we have that

$$|d\tilde{u}|_{HS(\bar{M},N)}^2 (x,t) = |d\tilde{u}|_{HS(\bar{M},N)}^2 (x,T) = \sum_{j=1}^{n-1} \langle d\tilde{u}(E_j), d\tilde{u}(E_j) \rangle_N (x,T)$$

$$= \sum_{j=1}^{n-1} \langle d\bar{u}(E_j), d\bar{u}(E_j) \rangle_N (x,T)$$

$$\leq \sum_{i=1}^{n} \langle d\bar{u}(E_i), d\bar{u}(E_i) \rangle_N (x,T)$$

$$= |d\bar{u}|_{HS(\bar{M},N)}^2 (x,T)$$

for all $(x,t) \in \bar{M}$. From (4) we thus obtain

$$(5) \qquad E^{\bar{M}}(\tilde{u}) = \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\tilde{u}|_{HS(\bar{M},N)}^2 (x,t) dV_M(x) \right) dt$$

$$\leq \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\bar{u}|_{HS(\bar{M},N)}^2 (x,T) dV_M(x) \right) dt$$

$$\leq \int_{\mathbb{T}} \left(\int_M e^{-f(x)} |d\bar{u}|_{HS(\bar{M},N)}^2 (x,t) dV_M(x) \right) dt = E^{\bar{M}}(\bar{u})$$

Now we prove that $\tilde{u}$ is homotopic to $\bar{u}$. To this end, we recall that since N is a $K(\pi,1)$–space, the homotopy class of a map $w : \bar{M} \to N$ is completely characterized by the action on $\pi_1(\bar{M})$ of the homeomorphism induced by w,

$$w_\sharp : \pi_1(\bar{M},(x,t)) \to \pi_1(N, w(x,t)).$$

We have that $\pi_1(\bar{M},(x,t)) \cong \pi_1(M,x) \times \pi_1(\mathbb{T},t)$. Since $\bar{u}$ is homotopic to $\bar{v}$ and, by construction, $\bar{v}_\sharp(\gamma) = id \in \pi_1(N)$ for all $\gamma \in \pi_1(\mathbb{T},t) < \pi_1(\bar{M},(x,t))$, we have that $\bar{u}_\sharp(\gamma) = id \in \pi_1(N)$ for all $\gamma \in \pi_1(\mathbb{T},t)$. On the other hand, by definition of $\tilde{u}$ we have that $\tilde{u}_\sharp(\gamma) = id \in \pi_1(N)$ for all $\gamma \in \pi_1(\mathbb{T},t)$ and $\tilde{u}_\sharp|_{\pi_1(M,x)}$ is conjugated to $\bar{u}_\sharp|_{\pi_1(M,x)}$. Accordingly, $\tilde{u}_\sharp$ is conjugated to $\bar{u}_\sharp$, so that $\tilde{u}$ is homotopic to $\bar{u}$. Since we have proved above that $\bar{u}$ minimizes the energy in its homotopy class, this together with (5) implies that $E^{\bar{M}}(\tilde{u}) = E^{\bar{M}}(\bar{u})$. Hence $\tilde{u}$ is a smooth minimizer of the energy in its homotopy class, it is thus harmonic and it has the aimed form.
Step c. We have shown that there exists a smooth map $u : M \to N$ such that $\tilde{u}(x,t) = u(x) = \bar{u}(x,T)$ for all $(x,t) \in \bar{M}$. This map u is a minimizer of the f–energy in its homotopy class. In fact, by contradiction suppose there exists another map $u_0 : M \to N$ homotopic to u with $E_f^M(u_0) < E_f^M(u)$. Then we can define $\tilde{u}_0 : \bar{M} \to N$ as $\tilde{u}_0(x,t) = u_0(x)$ for all $(x,t) \in \bar{M}$ and reasoning as above we would get that $\tilde{u}_0$ is homotopic to $\tilde{u}$ and $E^{\bar{M}}(\tilde{u}_0) < E^{\bar{M}}(\tilde{u})$. This would give the contradiction.

Hence u is a smooth minimizer of the f–energy in its homotopy class, and it is therefore an f–harmonic map. $\square$

To conclude this section, we would like to say some words to characterize the f–harmonic representative in homotopy class. In fact, thanks to the relations between M and $\bar{M}$ pointed out above, it is pretty easy to see how to generalize the uniqueness results obtained in the harmonic setting by Schoen and Yau [29] when M is parabolic (see also Remark 4 in [24] for the improved parabolic version). We recall that a manifold M is said to be f–parabolic if for some (hence every) compact set $K \subset M$ with non–empty interior the f–capacity of K is null, i.e.

$$\mathrm{Cap}_f(K) = \inf \{E_f(\varphi) \ : \ \varphi \in C_c^\infty(M), \ \varphi|_K \geq 1\} = 0.$$

There exist several equivalent definitions of f–parabolicity. To our purpose, let us just recall that M is f–parabolic if and only if every bounded f–subharmonic function is necessarily constant. This equivalence can be easily proved by adapting to the weighted setting the standard arguments used in the non–weighted case; see e.g. [13]. Now, consider a compact set $K \subset M$ with non–empty interior and define the compact set $\bar{K} = K \times \mathbb{T} \subset \bar{M}$. First, suppose $\mathrm{Cap}_f(K) = 0$ in M. Then by definition of f–capacity and by relation (3), it is clear that $\mathrm{Cap}(\bar{K}) = 0$ in $\bar{M}$. On the other hand, suppose $\bar{M}$ is parabolic. Let $\psi : M \to \mathbb{R}$ be a bounded f–subharmonic function. Defining $\bar{\psi} : \bar{M} \to \mathbb{R}$ as in Proposition 2.5, we have that $\bar{\psi}$ is a bounded subharmonic function on $\bar{M}$. Hence $\bar{\psi}$ and in turn ψ are constant. This proves the following

Lemma 2.6. *Let M be a complete Riemannian manifold and $f \in C^\infty(M)$. Then M is f–parabolic if and only if $\bar{M} = M \times_{e^{-f}} \mathbb{T}$ is parabolic.*

Now, suppose M is f–parabolic, N is non–positively curved and we are given two homotopic smooth f–harmonic maps $u, v : M \to N$ with finite f–energy. Defining maps $\bar{u}, \bar{v} : \bar{M} \to N$ as above we have that $\bar{u}$ and $\bar{v}$ are smooth finite–energy harmonic maps on the parabolic manifold $\bar{M}$, and they are homotopic since they are constantly extended on the fiber $\mathbb{T}$. Then we can apply Theorems 1 and 2 in [29] to $\bar{u}, \bar{v}$ to get

Theorem 2.7. *Let M and N be complete Riemannian manifolds and assume that M is f–parabolic,*

  i) *Let $u : M \to N$ be a smooth f–harmonic map of finite f–energy. If $^N\mathrm{Sect} < 0$, there's no other f–harmonic map of finite f–energy homotopic to u unless $u(M)$ is contained in a geodesic of N.*

  ii) *If $^N\mathrm{Sect} \leq 0$ and $u, v : M \to N$ are homotopic smooth f–harmonic maps of finite f–energy, then there is a continuous one–parameter family of maps $u_s : M \to N$ with $u_0 = u$ and $u_1 = v$ such that every u_s is a f–harmonic map of constant f–energy (independent of s) and for each $q \in M$ the curve $s \mapsto u_s(q)$, $s \in [0, 1]$, is a constant (independent of q) speed parametrization of a geodesic.*

The fact that the harmonic maps $\bar{u}_s : \bar{M} \to N$ obtained in the proof of Theorem 2 in [29] have the form $\bar{u}_s(x, t) := u_s(x)$ for all $(x, t) \in \bar{M}$ is not a direct consequence of the statement of Theorem 2 in [29], but it can be easily deduced by the construction given in Schoen and Yau's proof.

3. Vanishing results

A classical approach in harmonic maps theory leads to obtain vanishing results for the harmonic representative, imposing some additional assumptions on the curvature of M. In order to extend this part of the theory to our setting, we need a Bochner formula for f–harmonic maps. Such a formula is well known for functions [1], while for maps is essentialy contained in [15], Section 13. Since apparently no explicit version is given in the literature, we give a detailed proof in the following.

Proposition 3.1. *Let* $v : (M^m, \langle\,,\,\rangle_M) \to (N^n, \langle\,,\,\rangle_N)$ *be a* C^2 *map. Then*

$$(6) \quad \frac{1}{2}\Delta_f |dv|^2 = |Ddv|^2 + \langle dv, d\tau_f v\rangle_{HS} + \sum_{i=1}^m \left\langle dv({}^M\operatorname{Ric}_f(E_i, \cdot)^\sharp), dv(E_i)\right\rangle_N$$

$$- \sum_{i,j=1}^m \left\langle {}^N\operatorname{Riem}(dv(E_i), dv(E_j))dv(E_j), dv(E_i)\right\rangle_N,$$

where $\{E_i\}_{i=1}^m$ *is some chosen orthonormal frame on* M.

Remark 3.2. Defining the $(1,1)$–tensor field ${}^M\operatorname{rc}_f$ on M as

$$^M\operatorname{rc}_f(X) := {}^M\operatorname{Ric}_f(X, \cdot)^\sharp$$

for all vector fields X on M, we observe the following easy relation,

$$\sum_{i=1}^m \left\langle dv({}^M\operatorname{Ric}_f(E_i, \cdot)^\sharp), dv(E_i)\right\rangle_N = \operatorname{tr}_M \left\langle dv({}^M\operatorname{rc}_f(\cdot)), dv(\cdot)\right\rangle_N$$

$$= \left\langle dv({}^M\operatorname{rc}_f), dv\right\rangle_{HS}.$$

Proof. We start recalling the standard Bochner formula for the smooth map v, [9],

$$(7) \quad \frac{1}{2}\Delta |dv|^2 = |Ddv|^2 + \langle dv, d\tau v\rangle_{HS} + \sum_{i=1}^m \left\langle dv({}^M\operatorname{Ric}(E_i, \cdot)^\sharp), dv(E_i)\right\rangle_N$$

$$- \sum_{i,j=1}^m \left\langle {}^N\operatorname{Riem}(dv(E_i), dv(E_j))dv(E_j), dv(E_i)\right\rangle_N.$$

Inserting

$$\frac{1}{2}\Delta_f |dv|^2_{HS} = \frac{1}{2}\Delta |dv|^2_{HS} - \frac{1}{2}\nabla f(|dv|^2_{HS})$$

and

$$\langle dv, d\tau_f v\rangle_{HS} = \langle dv, d\tau v\rangle_{HS} - \langle dv, d(i_{\nabla f}dv)\rangle_{HS}$$

in (7) we get

$$\frac{1}{2}\Delta_f |dv|^2 = |Ddv|^2 + \langle dv, d\tau_f v\rangle_{HS} + \sum_{i=1}^m \left\langle dv({}^M\operatorname{Ric}(E_i, \cdot)^\sharp), dv(E_i)\right\rangle_N$$

$$+ \langle dv, d(i_{\nabla f}dv)\rangle_{HS} - \frac{1}{2}\nabla f(|dv|^2_{HS})$$

$$- \sum_{i,j=1}^m \left\langle {}^N\operatorname{Riem}(dv(E_i), dv(E_j))dv(E_j), dv(E_i)\right\rangle_N.$$

Hence (6) is proved once we show that

$$(8) \qquad \langle dv, d(i_{\nabla f}dv)\rangle_{HS} - \frac{1}{2}\nabla f(|dv|^2_{HS}) = \left\langle dv(\nabla_{(\cdot)}\nabla f), dv\right\rangle_{HS}.$$

Let $\{x^a\}_{a=1}^m$ be a local coordinate chart on M at $q \in M$ and $\{\theta^A\}_{A=1}^n$ and $\{E_A\}_{A=1}^n$ orthonormal coframe and dual frame on N at $v(q)$ respectively. Moreover denote the components of the metric on M as $\left\langle \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right\rangle_M =: g_{ab}$. We can write in coordinate

$$dv = v_a^A E_A \otimes dx^a, \qquad \text{and} \qquad \nabla f = f^a \frac{\partial}{\partial x^a}.$$

Then

$$i_{\nabla f} dv = v_a^A f^a E_A,$$

which gives

$$d(i_{\nabla f} dv) = \left(v_{ab}^A f^b + v_b^A f^b_{\ a} \right) E_A \otimes dx^a$$

and

$$(9) \qquad \langle dv, d(i_{\nabla f} dv) \rangle_{HS} = \sum_{A=1}^n v_d^A v_{ab}^A f^a g^{db} + \sum_{A=1}^n v_d^A v_a^A f^a_{\ b} g^{db}.$$

Moreover,

$$\nabla_{(\cdot)} \nabla f = \left(f^a_{\ b} + \Gamma^a_{bd} f^d \right) \frac{\partial}{\partial x^a} \otimes dx^b,$$

from which

$$dv(\nabla_{(\cdot)} \nabla f) = \left(v_a^A f^a_{\ b} + v_a^A \Gamma^a_{bd} f^d \right) E_A \otimes dx^b,$$

and

$$(10) \qquad \langle dv(\nabla_{(\cdot)} \nabla f), dv \rangle_{HS} = \sum_{A=1}^n v_a^A f^a_{\ b} v_d^A g^{bd} + v_a^A v_c^A \Gamma^a_{bd} f^d g^{bd}.$$

Here Γ denote the Christhoffel's symbols on M, ${}^M \nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} =: \Gamma^c_{ab} \frac{\partial}{\partial x^c}$. Finally,

$$\frac{1}{2} \nabla f (|dv|_{HS}^2) = \frac{1}{2} \sum_{A=1}^n \nabla f \left(\left\langle \left(\theta^A \circ dv \right)^\sharp, \left(\theta^A \circ dv \right)^\sharp \right\rangle_M \right)$$

$$= \sum_{A=1}^n \left\langle \nabla_{\nabla f} \left(\theta^A \circ dv \right)^\sharp, \left(\theta^A \circ dv \right)^\sharp \right\rangle_M.$$

Since

$$\left(\theta^A \circ dv \right)^\sharp = v_a^A g^{ab} \frac{\partial}{\partial x^b},$$

we get

$$\nabla_{\nabla f} \left(\theta^A \circ dv \right)^\sharp = \left(f^c \left(v_a^A g^{ab} \right)_c + f^a v_d^A g^{dc} \Gamma^b_{ac} \right) \frac{\partial}{\partial x^b},$$

and

$$(11) \qquad \frac{1}{2} \nabla f (|dv|_{HS}^2) = \sum_{A=1}^n v_b^A f^a v_d^A g^{dc} \Gamma^b_{ac} + f^a v_{da}^A g^{bd} v_b^A + f^a v_d^A v_b^A \left(\frac{\partial}{\partial x^a} g^{db} \right).$$

Combining (9), (10) and (11) we get that (8) is proved provided

$$(12) \qquad 2 \sum_{A=1}^n v_b^A f^a v_d^A g^{dc} \Gamma^b_{ac} + f^a v_d^A v_b^A \left(\frac{\partial}{\partial x^a} g^{db} \right) = 0.$$

Now, since the Levi–Civita connection is compatible with the metric,

$$\frac{\partial}{\partial x^a} g_{bc} = g_{dc} \Gamma^d_{ba} + g_{db} \Gamma^d_{ca}$$

and

$$\left(\frac{\partial}{\partial x^a}g^{bc}\right)g_{cd} = -g^{bc}\left(\frac{\partial}{\partial x^a}g_{cd}\right),$$

from which

$$\frac{\partial}{\partial x^a}g^{bd} = -g^{bc}\Gamma_{ca}^d - g^{cd}\Gamma_{ca}^b$$

and, in turn,

$$\sum_{A=1}^{n} f^a \left[2v_b^A v_d^A g^{dc}\Gamma_{ac}^b + v_d^A v_b^A \left(\frac{\partial}{\partial x^a}g^{db}\right)\right]$$
$$= \sum_{A=1}^{n} f^a \left[2v_b^A v_d^A g^{dc}\Gamma_{ac}^b - v_b^A v_d^A g^{bc}\Gamma_{ac}^d - v_b^A v_d^A g^{dc}\Gamma_{ac}^b\right] = 0.$$

This latter proves (12) and concludes the proof. □

We are now ready to prove the following vanishing result.

Theorem 3.3. *Let M^m, N^n be complete Riemannian manifolds and $f \in C^\infty(M, \mathbb{R})$. Assume that $^M \mathrm{Ric}_f \geq 0$ and $^N \mathrm{Sect} \leq 0$. Consider an f–harmonic map $v : M \to N$ with finite f–energy $E_f(v) < +\infty$. Then v is constant provided at least one of the following assumption is satisfied*

- (b) *there exists a constant $C > 0$ such that $|f| \leq C$;*
- (c) *f is convex and the set of its critical points is unbounded;*
- (d) *$\mathrm{Vol}_f(M) := \int_M e^{-f}dV_M = +\infty$;*
- (e) *there is a point $q_0 \in M$ such that $^M \mathrm{Ric}_f|_{q_0} > 0$;*
- (f) *there is a point $q_1 \in M$ such that $^M \mathrm{Ric}(X, X)|_{q_1} \neq 0$ for all $0 \neq X \in T_{q_1}M$.*

Moreover if

- (g) *$^N \mathrm{Sect} < 0$,*

then either v is constant or the whole image $v(M)$ is contained in a geodesic of N.

Remark 3.4. The case of M compact is in [15], p. 367. See also [28] for the non–compact harmonic case.

We partially follow the proof given in [27] for solutions of generic Bochner–type inequalities.

Proof. Set $\phi := |dv|_{HS}$ and $G(v) := |Ddv|^2 - |\nabla|dv||^2$. Notice that

$$\frac{1}{2}\Delta_f\phi^2 = \phi\Delta_f\phi + |\nabla\phi|^2.$$

Since v is f–harmonic, by the Bochner formula (6) and the Kato's inequality we get

(13) $$\phi\Delta_f\phi \geq G(v) \geq 0.$$

Let $\rho \in C_c^\infty(M)$ to be chosen later. Then

$$\int_M \rho^2\phi\Delta_f\phi e^{-f}dV_M \geq \int_M \rho^2 G(v)e^{-f}dV_M \geq 0.$$

Stokes' Theorem and Young's inequality yield

$$\int_M \rho^2 \phi \Delta_f \phi e^{-f} dV_M$$

$$= \int_M \rho^2 \phi \operatorname{div}(e^{-f} \nabla \phi) dV_M$$

$$= -\int_M 2\rho\phi \langle \nabla\rho, \nabla\phi \rangle_M e^{-f} dV_M - \int_M \rho^2 |\nabla\phi|^2 e^{-f} dV_M$$

$$\leq \epsilon^{-1} \int_M \phi^2 |\nabla\rho|^2 e^{-f} dV_M - (1-\epsilon) \int_M \rho^2 |\nabla\phi|^2 e^{-f} dV_M$$

for any $0 < \epsilon < 1$. From this latter we get

$$(14) \qquad 0 \leq (1-\epsilon) \int_M \rho^2 |\nabla\phi|^2 e^{-f} dV_M + \int_M \rho^2 G(v) e^{-f} dV_M$$

$$\leq \epsilon^{-1} \int_M \phi^2 |\nabla\rho|^2 e^{-f} dV_M.$$

Choose the smooth cut–off $\rho = \rho_R$ s.t. $\rho \leq 1$ on M, $\rho|_{B_R} \equiv 1$, $\rho|_{M \setminus B_{2R}} \equiv 0$ and $|\nabla\rho| \leq 2/R$. Replacing $\rho = \rho_R$ in (14) we obtain

$$(15) \qquad 0 \leq (1-\epsilon) \int_{B_R} |\nabla\phi|^2 e^{-f} dV_M + \int_{B_R} G(v) e^{-f} dV_M$$

$$\leq \frac{4\epsilon^{-1}}{R^2} \int_{B_{2R}} \phi^2 e^{-f} dV_M.$$

By the assumption $E_f(v) < +\infty$ we can let $R \to \infty$ applying monotone convergence at RHS to get $\nabla\phi \equiv 0$, i.e. $\phi = |dv| \equiv const.$, and

$$(16) \qquad 0 \equiv G(v) = |Ddv|^2 - |\nabla\phi|^2 = |Ddv|^2.$$

Suppose $|dv| \equiv C > 0$. Then the finiteness of the f–energy of v gives that $\operatorname{Vol}_f(M) < +\infty$. If either $|f|$ is uniformly bounded or f is convex and the set of its critical points is unbounded, then Theorem 1.3 and 5.3 in [33] implies that M has at least linear f–volume growth, giving a contradiction.

In general we have $Ddv \equiv 0$, i.e. $v : M \to N$ is totally geodesic, which in turn gives that v is harmonic, i.e. $\tau v = 0$ and

$$(17) \qquad i_{\nabla f} dv = dv(\nabla f) = \tau_f v - \tau v \equiv 0.$$

Accordingly, the Bochner formula (6) reads

$$(18) \qquad 0 = \sum_{i=1}^m \left\langle dv({}^M \operatorname{Ric}_f(E_i, \cdot)^\sharp), dv(E_i) \right\rangle_N$$

$$- \sum_{i,j=1}^m \left\langle {}^N \operatorname{Riem}(dv(E_i), dv(E_j)) dv(E_j), dv(E_i) \right\rangle_N,$$

and by the curvature sign assumptions both

$$(19) \qquad \sum_{i=1}^m \left\langle dv({}^M \operatorname{Ric}(E_i, \cdot)^\sharp), dv(E_i) \right\rangle_N + \sum_{i=1}^m \left\langle dv({}^M \nabla_{E_i} \nabla f), dv(E_i) \right\rangle_N = 0$$

and

$$\sum_{i,j=1}^m \left\langle {}^N \operatorname{Riem}(dv(E_i), dv(E_j)) dv(E_j), dv(E_i) \right\rangle_N = 0.$$

First, suppose that $^N \mathrm{Sect} < 0$, then $dv(E_i) \parallel dv(E_j)$ for all $i, j = 1, \ldots, n$ and we conclude that $v(M)$ must be contained in a geodesic of N.

On the other hand, suppose that $^M \mathrm{Ric}_f \, |_{q_0} > 0$ at some point $q_0 \in M$. Then necessarily $dv(q) = 0$ which gives $dv \equiv 0$.

Moreover, since

$$0 = Ddv(X, Y) = (D_Y dv)(X) = {}^N \nabla_{dv(Y)} dv(X) - dv(^M \nabla_Y X)$$

for all X, Y vector fields on M, (17) implies

$$\left\langle dv(^M \nabla_{E_i} \nabla f), dv(E_i) \right\rangle_N = \left\langle Ddv(\nabla f, E_i), dv(E_i) \right\rangle_N$$
$$= \left\langle ^N \nabla_{dv(E_i)} dv(\nabla f), dv(E_i) \right\rangle_N = 0$$

for each $i = 1, \ldots, m$. Since $^M \mathrm{Ric}_f \geq 0$, (19) in particular gives

$$\left\langle dv(^M \mathrm{Ric}(E_i, \cdot)^\sharp), dv(E_i) \right\rangle_N = 0$$

for each $i = 1, \ldots, m$. Hence, if there exists some point $q_1 \in M$ such that $^M \mathrm{Ric}(X, X)|_{q_1} \neq 0$ for all vector $0 \neq X \in T_{q_1} M$, then v is once again necessarily constant.   □

Remark 3.5. Theorem 3.3 holds also in the more general assumption

$$\int_{B_R} |dv|^2 e^{-f} dV_M = o(R),$$

instead of $E_f(v) < +\infty$.

Remark 3.6. One could combine Proposition 2.4 or Proposition 2.5 above with the vanishing result in the harmonic case to obtain directly a vanishing result for *f*–harmonic maps. Nevertheless in this case the assumptions on $^M \mathrm{Ric}$ are more involved and less natural in view of the applications to weighted manifolds and gradient Ricci solitons. Namely, one has that

$$\widetilde{^M} \mathrm{Ric} = {}^M \mathrm{Ric} + \left[\mathrm{Hess}\, f + \frac{df \otimes df}{m-2} \right] + \frac{1}{m-2} \left[\Delta f + |df|^2 \right] \langle , \rangle_M$$

where $\widetilde{M} = (M, e^{-\frac{2f}{m-2}} \langle , \rangle_M)$, while

$$^{\bar{M}} \mathrm{Ric}(X, X) = {}^M \mathrm{Ric}(X, X) + \mathrm{Hess}\, f(X, X) - <X, \nabla f>^2,$$

for $X \in TM < T\bar{M}$, and

$$^{\bar{M}} \mathrm{Ric}(\nu, \nu) = \Delta f - |df|^2 = \Delta_f f,$$

for $\nu \in T\mathbb{T} < T\bar{M}$.

Remark 3.7. The statement of Theorem 3.3 is non–trivial, as shown in the following example.

Let $\mathbb{T}^n$ be the standard flat n–torus $\mathbb{R}^n / \mathbb{Z}^n$. For the easiness of notation, here we parameterize the torus as $\mathbb{T}^n = [-1, 1]^n$ with the usual identifications on the boundaries. Consider a map $v : \mathbb{R}^2 \times \mathbb{T} = M \to N = \mathbb{T}^3$ and a function $f : \mathbb{R}^2 \times \mathbb{T} = M \to \mathbb{R}$ given by $v(x, y, [z]) := ([0], [0], [z])$ and $f(x, y, [z]) := \frac{x^2 + y^2}{2}$. Since $dv = dz \otimes \frac{\partial}{\partial z}$ is parallel, we have in particular that $\tau v = 0$. Moreover $df = x dx + y dy$ implies $\tau_f v = dv(\nabla f) = 0$. As a consequence v is a non–constant *f*–harmonic map (notably in a non–trivial homotopy class of maps), and in fact none of the assumptions (b) to (f) of Theorem 3.3 is satisfied. In particular, (b) *f*

is upper unbouded, (c) f is convex but with a bounded set of critical points, (d) the f–volume of M satisfies

$$\mathrm{Vol}_f(\mathbb{R}^2 \times \mathbb{T}) = \int_{-1}^{1} \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dxdydz = 4\pi < +\infty,$$

(e) $^M \mathrm{Ric}_f(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) \equiv 0$ and (f) $^M \mathrm{Ric} \equiv 0$.

Remark 3.8. A similar example shows that the weaker assumption suggested in Remark 3.5 is sharp. In particular let M, N and v be as in Remark 3.7 but now choose $f(x, y, [z]) = \frac{x^2}{2}$. With these choices, $^M \mathrm{Ric}_f(M) \geq 0$ and assumption (c) of Theorem 3.3 is satisfied, but the f–volumes of geodesic balls have exactly linear growth and in fact v is once again a non–constant f–harmonic map.

Combining Theorem 2.1 and Theorem 3.3 we get the following result, which corresponds to the $k \equiv 0$ case of Theorem A.

Theorem 3.9. *Let M be a complete Riemannian manifold and N a compact Riemannian manifold with $^N \mathrm{Sect} \leq 0$. Let $f \in C^\infty(M)$ and suppose $^M \mathrm{Ric}_f \geq 0$. Consider a continuous map $u : M \to N$ with finite f–energy $E_f(u) < +\infty$. Then u is homotopic to a constant provided at least one of the assumptions (b) to (f) of Theorem 3.3 is satisfied.*
On the other hand, if we assume that $^N \mathrm{Sect} < 0$, then u is homotopic either to a constant or to a totally geodesic map whose image is contained in a geodesic of N.

4. Spectral assumptions

In this section we suppose that the underlying manifold M satisfies

$$^M \mathrm{Ric}_f \geq -k^2$$

for some function $k(x)$, provided

(20) $$\lambda_1(-\Delta_f - Hk^2) \geq 0$$

for some $H > 1$. As pointed out in Remark 1.1, this corresponds to ask that

$$\int_M |\nabla\varphi|^2 e^{-f} dV_M - \int_M Hk^2\varphi^2 e^{-f} dV_M \geq 0$$

for all $\varphi \in C_c^\infty(M)$. On the other hand, it was observed in [31, 2] that (20) is also equivalent to ask that

$$\lambda_1\left(-\Delta - \left(\frac{1}{2}\Delta f - \frac{1}{4}|\nabla f|^2 + Hk^2\right)\right) \geq 0.$$

Proceeding as in [27], we can show that Theorem 3.3 and Theorem 3.9 hold in these more general assumptions. The result thus obtained, together with Theorem 3.9, will complete the proof of Theorem A.

Theorem 4.1. *Let M^m, N^n be complete Riemannian manifolds and $f \in C^\infty(M, \mathbb{R})$. Assume that $^M \mathrm{Ric}_f \geq -k^2$ for some function $k(x) \not\equiv 0$ satisfying (20) and that $^N \mathrm{Sect} \leq 0$. Then the following hold*

(i) *Any smooth f–harmonic map $v : M \to N$ with finite f–energy $E_f(v) < +\infty$ is constant.*

(ii) *If N is compact, then any continuous map $u : M \to N$ with finite f–energy $E_f(u) < +\infty$ is homotopic to a constant.*

Proof. First we note that, thanks to Theorem 2.1, (i) trivially implies (ii).
To prove (i), we define objects as in the proof of Theorem 3.3. Choosing $\varphi = \rho\phi$,
(2) yields that

$$\int_M k^2\rho^2\phi^2 e^{-f}dV_M \leq (1+\delta)H^{-1}\int_M \rho^2|\nabla\phi|^2 e^{-f}dV_M$$
$$+ (1+\delta^{-1})H^{-1}\int_M \phi^2|\nabla\rho|^2 e^{-f}dV_M,$$

for any fixed $\delta > 0$. Proceeding as in the proof of Theorem 3.3 we note that
inequalities (13) and (14) are respectively replaced by

$$\phi\Delta_f\phi \geq G(v) - k^2\phi^2 \geq 0.$$

and

$$0 \leq (1 - \epsilon - H^{-1}(1+\delta))\int_M \rho^2|\nabla\phi|^2 e^{-f}dV_M + \int_M \rho^2 G(v)e^{-f}dV_M$$
$$\leq (\epsilon^{-1} + H^{-1}(1+\delta^{-1}))\int_M \phi^2|\nabla\rho|^2 e^{-f}dV_M,$$

up to choose δ small enough. From now on, the proof is the same as for the case
$k \equiv 0$. We have only to remark that the spectral assumption implies that M has
infinite volume, as observed in Remark 1.8 (b) in [22]. In fact, by contradiction,
if $\mathrm{Vol}_f(M) < +\infty$ (more generally if it has at most linear growth), then in (2)
we can choose $\varphi = \varphi_R$ to be a family of cut–offs such that $\varphi_R \leq 1$, $\varphi_R|_{B_R} \equiv 1$,
$\varphi_R|_{M\setminus B_{2R}} \equiv 0$ and $|\nabla\varphi_R| \leq 2/R$, and letting $R \to \infty$ we obtain that $k \equiv 0$.   □

Remark 4.2. With respect to the standard $f = 0$ case, here we need $H > 1$
because no refined Kato inequality is given for f–harmonic maps. We recall that
for a smooth harmonic map v, the refined Kato inequality is the relation, [3, 5]

$$|Ddv|^2 - |\nabla|dv||^2 \geq K|\nabla|dv||^2$$

holding with $K = 1/(m-1)$. As a matter of fact, it turns out that such a constant
$K > 0$ can not exist for general f–harmonic maps. To see this, consider functions
$v, f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$v(x,y,z) := \int_0^x e^{-\frac{t^2}{2}}dt, \qquad f(x,y,z) = -\frac{x^2}{2}.$$

We have that

$$\Delta v = \frac{\partial^2 v}{\partial x^2} = -xe^{-\frac{x^2}{2}}$$

and

$$\langle \nabla f, \nabla v\rangle = \left\langle -x\frac{\partial}{\partial x}, e^{-\frac{x^2}{2}}\frac{\partial}{\partial x}\right\rangle = -xe^{-\frac{x^2}{2}},$$

so that

$$\tau_f v = \Delta v - \langle\nabla f, \nabla v\rangle = 0.$$

On the other hand $d|\nabla v| = -xe^{-\frac{x^2}{2}}dx$ and $Ddv = -xe^{-\frac{x^2}{2}}dx^2$, which implies

$$|Ddv|^2 \equiv |\nabla|dv||^2.$$

5. Geometric context and applications

The importance of studying topological rigidity properties of weighted manifolds under curvature restrictions arises from the need to understand the topology of gradient Ricci solitons.

Concerning the shrinking side, i.e. $\mathrm{Ric}_f = \lambda \langle \, , \, \rangle$ for some constant $\lambda > 0$, we mention that it is not difficult to see that the full conclusion of the classical Myers–Bonnet theorem cannot be extended to Ric_f. Indeed the Gaussian space $(\mathbb{R}^m, \langle \, , \, \rangle_{can}, e^{-|x|^2/2}dV_{\mathbb{R}^m})$ is a non–compact, complete weighted manifold with $\mathrm{Ric}_f = 1 > 0$. In order to recover compactness we have to impose, besides the positive constant lower bound on Ric_f, further conditions on the growth of f or on its gradient, see [12, 11]. Nevertheless, as initially investigated in works of M. Fernández–López and E. García–Río in the compact case, [12], and later in the complete non–compact case by W. Wylie, [35], a close relationship beetween Ric_f and the fundamental group of a weighted manifold still survive. Namely, Myers–type results in this contest establish the finiteness of the fundamental group if $\mathrm{Ric}_f \geq c^2 > 0$ (and in particular for shrinking gradient Ricci solitons). Let us also mention that very recently in [22] these results were extended in the direction of the classical Ambrose theorem to complete weighted manifolds $(M^m, \langle \, , \, \rangle_M, e^{-f}dV_M)$ such that, fixed a point $o \in M$, for every unit speed geodesic γ issuing from $\gamma(0) = o$ we have

$$(i) \; {}^M \mathrm{Ric}_f(\dot{\gamma}, \dot{\gamma}) \geq \mu \circ \gamma + \langle \nabla g \circ \gamma, \dot{\gamma} \rangle_M, \qquad (ii) \; \int_0^{+\infty} \mu \circ \gamma(t)dt = +\infty,$$

fore some functions $\mu \geq 0$ and g bounded. For more details see Theorem 9.1 in [22].

We now come to analyze the topology of weighted manifolds $(M^m, \langle \, , \, \rangle_M, e^{-f}dV_M)$ with ${}^M \mathrm{Ric}_f \geq 0$. Altough we are interested in particular to the study of topological properties "in the small" we mention that very recently some results regarding the topology at infinity of weighted manifolds with ${}^M \mathrm{Ric}_f \geq 0$ and in particular of gradient steady Ricci solitons were obtained by O. Munteanu and J. Wang in [19]. In particular, they prove that gradient steady Ricci solitons are either connected at infinity or they are isometric to a Ricci flat cylinder.

The following theorem encloses some known topological results in case ${}^M \mathrm{Ric}_f \geq 0$ which are particularly relevant to our investigation. These are obtained in [16, 33, 36] and adapting to weighted setting a result in [30]. Note that, in particular, the first part of conclusion (ii) and conclusions (iii) and (iv) apply to steady gradient Ricci solitons, i.e. when ${}^M \mathrm{Ric}_f \equiv 0$.

Theorem 5.1. *Let $(M^m, \langle \, , \, \rangle_M, e^{-f}dV_M)$ be an m–dimensional complete weighted manifold, then the following hold:*

    (i) *if M is compact, ${}^M \mathrm{Ric}_f \geq 0$ and ${}^M \mathrm{Ric}_f > 0$ at one point then $|\pi_1(M)| < \infty$;*

    (ii) *if M is non–compact, ${}^M \mathrm{Ric}_f \geq 0$, and $|f|$ is bounded, then M either satisfies the loops to infinity property or has a double covering which splits. In particular if ${}^M \mathrm{Ric}_f > 0$ then M satisfies the loops to infinity property;*

    (iii) *if M is non–compact with ${}^M \mathrm{Ric}_f \geq 0$, $|f|$ is bounded and $D \subset M$ is a pre-compact set with simply connected boundary, then $\pi_1(D)$ can only contain elements of order 2.*

(iv) *if M is non–compact and $^M\mathrm{Ric}_f \geq 0$, and either f is a convex function and attains its minimum or $|f|$ is bounded, then $b_1(M) \leq m$.*

As a first application of their vanishing result for the harmonic representative, R. Schoen and S.–T. Yau, in [28], studied the topology of manifolds with non–negative Ricci curvature. We can now naturally generalize their work to the case of manifolds with non–negative Ric_f curvature. Contrary to all the results contained in Theorem 5.1, the technique used here can be extended, under the validity of the spectral assumption (1) on Δ_f, also to the more general case $^M\mathrm{Ric}_f \geq -k^2$. In particular as a consequence of Theorem A we obtain Corollary B, which predicts information on the topology of compact domains with simply connected boundary in a complete manifold M. While Corollary B (i) is a simple generalization of the non–weighted results, the conclusion (ii) is new. In fact for $f \equiv 0$, at least one of the conditions (a) or (c) of Theorem A is automatically satisfied, so that (ii) is trivially contained in (i). Accordingly, in order to prove Corollary B in the weighted setting, some further work is necessary. In particular we need the following result.

Theorem 5.2. *Let M and N be complete Riemannian manifolds, $u \in C^2(M, N)$ such that $Ddu = 0$ and $\mathrm{rk}(u) \equiv 1$. Then $u(M) \subseteq \gamma$, γ geodetic of N and*

(1) *if γ is closed $u_\sharp(\pi_1(M, x_0)) \leq \left\langle [\gamma]_{u(x_0)} \right\rangle \leq \pi_1(N, u(x_0))$;*

(2) *if γ is not closed then $u_\sharp(\pi_1(M, x_0)) = \mathbf{1}_{\pi_1(N, u(x_0))}$.*

Proof. Since u is a totally geodesic map of rank 1, by standard arguments we know that there exists a geodesic γ of N such that $u(M) \subset \gamma$. Without loss of generality, we can take a constant speed parametrization $\gamma : \mathbb{R} \to N$. Fix an element $\mathfrak{g} \in \pi_1(M, x_0)$. In case γ is closed, hence periodic so that up to a linear change of variable it can be reparametrized as $\gamma : \mathbb{T} \to N$, we want to show that

$$[u \circ \sigma]_{u(x_0)} = [\gamma]_{u(x_0)}^l \tag{21}$$

for some $l \in \mathbb{Z}$ and for some (hence any) continuous loop $\sigma : [0, 1] \to M$ based at x_0, i.e. $\sigma(0) = \sigma(1) = x_0$, such that $[\sigma]_{x_0} = \mathfrak{g}$. On the other hand, in case $\gamma : \mathbb{R} \to N$ is not closed, i.e. non–periodic, the thesis corresponds to prove that

$$[u \circ \sigma]_{u(x_0)} = \mathbf{1}_{\pi_1(N, u(x_0))}. \tag{22}$$

First, we show that we can choose the continuous loop $\sigma : [0, 1] \to M$ based at x_0, $[\sigma]_{x_0} = \mathfrak{g}$, in such a way that $\sigma|_{(0,1)}$ is a constant speed geodesic. To this end, let $\tilde{M}$ be the universal cover of M and $P_M : \tilde{M} \to M$ the covering projection. Fix an element $\tilde{x}_0$ in the fiber $P_M^{-1}(x_0)$. Then we can lift σ to a continuous path $\tilde{\sigma} : [0, 1] \to \tilde{M}$ which satisfies

$$P_M \circ \tilde{\sigma}(t) = \sigma(t), \qquad \tilde{\sigma}(0) = \tilde{x}_0, \qquad \tilde{\sigma}(1) = \tilde{x}_0'$$

for all $t \in [0, 1]$ and for some $\tilde{x}_0' \in P_M^{-1}(x_0)$ (possibly $\tilde{x}_0 = \tilde{x}_0'$). Consider a constant speed (possibly constant) geodesic $\tilde{\nu} : [0, 1] \to \tilde{M}$ joining $\tilde{x}_0$ to $\tilde{x}_0'$. Since $\tilde{M}$ is simply connected there exists a homotopy $\tilde{H}$ relative to $\{\tilde{x}_0, \tilde{x}_0'\}$ deforming $\tilde{\sigma}$ into $\tilde{\nu}$, i.e. $\tilde{H} \in C^0([0, 1]^2, \tilde{M})$ and $\tilde{H}(0, t) = \tilde{\sigma}(t)$, $\tilde{H}(1, t) = \tilde{\nu}(t)$, $\tilde{H}(s, 0) = \tilde{x}_0$ and $\tilde{H}(s, 1) = \tilde{x}_0'$ for all $s, t \in [0, 1]$. Hence we can project $\tilde{H}$ to the homotopy $H := P_M \circ \tilde{H} \in C^0([0, 1]^2, M)$, so that the loop ν based at x_0 defined by $\nu(t) := H(1, t) = P_M(\tilde{\nu}(t))$ satisfies $[\nu]_{x_0} = [\sigma]_{x_0} = \mathfrak{g}$ and, since covering projection maps are local isometries, $\nu|_{(0,1)}$ is a geodesic.

Now, if $u \circ \nu$ is constant, then clearly $[u \circ \nu]_{u(x_0)} = \mathbf{1}_{\pi_1(N, u(x_0))}$, so that $u_\sharp([\nu]_x) =$

$\mathbf{1}_{\pi_1(N,u(x))}$.

On the other hand, suppose that $u \circ \nu \in C^0([0,1], N)$ is not constant. Then, since $Ddu = 0$, $u \circ \nu$ is a non–trivial constant speed geodesic arc in N, i.e. ${}^N\nabla_{du(\dot\nu)}du(\dot\nu) \equiv 0$, and $u \circ \nu \subset \gamma$. Since $\nu(0) = \nu(1)$, ν can be seen as a continuous function on $\mathbb{T} = [0,1]/ \sim$ smooth in $(0,1)$. Moreover, since $Ddu = 0$ and $\mathrm{rk}(u) \equiv 1$, we have that

$$du(\dot\nu(1)) \parallel du(\dot\nu(0)) \qquad \text{and} \qquad |du(\dot\nu(1))| = |du(\dot\nu(0))| \neq 0$$

in $T_{u(x_0)}N$, and, since $u \circ \nu|_{(0,1)}$ is a non–trivial geodesic of N, necessarily it is

$$du(\dot\nu(1)) = du(\dot\nu(0)).$$

Then both γ and $u \circ \nu$ are nontrivial closed geodesic of N, and $u \circ \nu = \gamma^l$ (i.e. γ covered l times) for some $l \in \mathbb{Z}$. In particular we get $u_\sharp([\nu]_x) = [\gamma]^l_{u(x)}$.

Since we have arbitrarily chosen the element $\mathfrak{g} \in \pi_1(M, x_0)$, this permits to conclude. $\square$

As a consequence of Theorem 5.2 and Theorem A we obtain the desired topological result.

Proof (of Corollary B). We give just an idea of the proof. For the details see the proof of Theorem 6.21 in [25].

Consider a homomorphism $\sigma \in \mathrm{Hom}(\pi_1(D), \pi_1(N))$. Since N is $K(\pi,1)$, according to the theory of aspherical spaces there exists a map $\hat{u} : D \to N$ such that $\sigma = \alpha \circ \hat{u}_\sharp$ for some automorphism $\alpha \in \mathrm{Aut}(\pi_1(N))$. Since ∂D is simply connected, $\hat{u}$ can be extended to a map $u : M \to N$ such that $u|_{M \setminus D'}$ is constant for some compact set $D \subset\subset D' \subset\subset M$. Then u has finite f–energy and we can apply Theorem A. First, if one of the condition (a) to (f) is satisfied, we deduce that u is homotopic to a constant and accordingly the homomorphism σ is trivial. On the other hand, if ${}^N\mathrm{Sect} < 0$, then Theorem A says that u is a totally geodesic map of $\mathrm{rk}(u) \equiv 1$ so that $u(M)$ is contained in some geodesic γ of N. Hence, an application of Theorem 5.2 yields that, for some x_0 in D,

$$\hat{u}_\sharp(\pi_1(D, x_0)) = u_\sharp(\pi_1(D, x_0)) < u_\sharp(\pi_1(M, x_0))$$

$$< \left\langle [\gamma]_{u(x_0)} \right\rangle < \pi_1(N, u(x_0)).$$

$\square$

Remark 5.3. When M is compact and (e) holds or M is complete non–compact and (b) is satisfied, the conclusion of Corollary B (i) follows from Theorem 5.1 (i) and (iii). Indeed, let $\rho : \pi_1(M) \to \pi_1(N)$ be a non–trivial homomorphism of $\pi_1(M)$ into the fundamental group of a compact manifold N with ${}^N\mathrm{Sect} \leq 0$. Then every $\mathfrak{g} \in \pi_1(M)$, and hence $\rho(\mathfrak{g})$, must have finite order. On the other hand by Cartan theorem all non–trivial elements of the fundamental group of a complete Riemannian manifold of non–positive curvature have infinite order. Contradiction.

Let us now consider the case where $(M^m, \langle\,,\,\rangle_M, e^{-f}dV_M)$ supports an expanding gradient Ricci soliton structure. Very recently in [18] it has been shown that a non–trivial expanding Ricci soliton must be connected at infinity provided its scalar curvature satisfies a suitable lower bound. Starting from Corollary B we can go a step further in the understanding of the topology of this class of manifolds. Indeed, in [18] the following lemma is proved. Letting ${}^MS$ denote the scalar curvature of M,

recall that by the scalar curvature estimates proved for expanding Ricci solitons, $m\lambda \leq \inf_M {}^M S \leq 0$ and ${}^M S(x) > m\lambda$, unless M is Einstein and the soliton is trivial (i.e. f is constant); see [26].

Lemma 5.4 (Lemma 5.3 in [18]). *Let $(M^m, \langle\,,\,\rangle_M, \nabla f)$ be a complete non–trivial expanding gradient Ricci soliton. Define $\rho := {}^M S - m\lambda$. Then $\rho > 0$ and*

$$\int_M \rho \varphi^2 e^{-f} dV_M \leq \int_M |\nabla \varphi|^2 e^{-f} dV_M,$$

for any $\varphi \in C_0^\infty$. In particular $\lambda_1(-\Delta_f - \rho) \geq 0$.

Remark 5.5. As a consequence of Lemma 5.4 and Corollary 1.7 in [22] we observe incidentally that for any complete non–trivial expanding Ricci soliton with scalar curvature such that ${}^M S - m\lambda > c > 0$ the quantity $\int_{B_r} |\nabla f|^p e^{-f} dV_M$ has to grow at least quadratically, for any $p > 1$.

The lower bound for the bottom of the spectrum of Δ_f obtained in Lemma 5.4 permits to obtain conditions on the scalar curvature of an expanding gradient Ricci soliton in order to guarantee condition (1) for some $H > 1$, where $k \not\equiv 0$, and hence to apply Corollary B under the assumption (a). Indeed, in order to conclude the proof of Theorem C observe that if $S > (m-1)\lambda$ and $H > 1$,

$$\int_M ({}^M S - m\lambda)\varphi^2 e^{-f} dV_M \geq -\int_M H\lambda \varphi^2 e^{-f} dV_M$$

for any $\varphi \in C_0^\infty$. By Lemma 5.4 this permits to deduce (2) with $k^2 = -\lambda$.

References

1. D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, pp. 177–206.
2. G. P. Bessa, S. Pigola, and A.G. Setti, *Spectral and stochastic properties of the f–laplacian, solutions of PDE's at infinity and geometric applications*, arXiv:1107.1172v1 (2011).
3. T. Branson, *Kato constants in Riemannian geometry*, Math. Res. Lett. **7** (2000), no. 2-3, 245–261.
4. F. E. Burstall, *Harmonic maps of finite energy from noncompact manifolds*, J. London Math. Soc. (2) **30** (1984), no. 2, 361–370.
5. D. M. J. Calderbank, P. Gauduchon, and M. Herzlich, *Refined Kato inequalities and conformal weights in Riemannian geometry*, J. Funct. Anal. **173** (2000), no. 1, 214–255.
6. N. Course, *f–harmonic maps*, Ph.D Thesis (2004), University of Warwick.
7. ———, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. **13** (2007), 423–435 (electronic).
8. J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), no. 1, 1–68.
9. ———, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, vol. 50, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1983.
10. J. Eells and J.H. Sampson, *Harmonic mappings of riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
11. M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons: the equation point of view*, Manuscripta Math. **127** (2008), no. 3, 345–367.
12. M. Fernández-López and E. García-Río, *A remark on compact Ricci solitons*, Math. Ann. **340** (2008), no. 4, 893–896.
13. A. A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the brownian motion on riemannian manifolds*, Bull. of the Amer. Math. Soc. **36** (1999), no. 2, 135–249.
14. P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967), 673–687.

15. A. Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69), Academic Press, London, 1968/1969, pp. 341–402.

16. ———, *Variétés riemanniennes à tenseur C non négatif*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A650–A653.

17. F. Morgan, *Myers' theorem with density*, Kodai Math. J. **29** (2006), no. 3, 455–461.

18. O. Munteanu and J. Wang, *Analysis of weighted Laplacian and applications to Ricci solitons.* To appear on Comm. Anal. Geom., arXiv:1112.3027v1.

19. ———, *Smooth metric measure spaces with nonnegative curvature*, Comm. Anal. Geom **19** (2011), no. 3, 451–486.

20. B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, With applications to relativity.

21. Y.-L. Ou, *On f-harmonic morphisms between Riemannian manifolds*, arXiv:1103.5687v1 (2011).

22. S. Pigola, M. Rigoli, M. Rimoldi, and A.G. Setti, *Ricci almost solitons.* To appear on Ann. Sc. Norm. Super. Pisa Cl. Sci., arXiv:1003.2945v1 (2010).

23. S. Pigola, M. Rigoli, and A. G. Setti, *Vanishing theorems on Riemannian manifolds, and geometric applications*, J. Funct. Anal. **229** (2005), no. 2, 424–461.

24. ———, *Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds*, Math. Z. **258** (2008), 347–362.

25. ———, *Vanishing and finiteness results in geometric analysis*, Progress in Mathematics, vol. 266, Birkhäuser Verlag, Basel, 2008, A generalization of the Bochner technique.

26. S. Pigola, M. Rimoldi, and A.G. Setti, *Remarks on non-compact gradient Ricci solitons*, Math. Z. **268** (2011), 777–790.

27. S. Pigola and G. Veronelli, *Remarks on L^p–vanishing results in geometric analysis.* To appear on Intern. J. of Math., Online First: DOI 10.1142/S0129167X11007513 , arXiv:1011.5413v2 (2010).

28. R. Schoen and S.-T. Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature*, Comment. Math. Helv. **51** (1976), no. 3, 333–341.

29. ———, *Compact group actions and the topology of manifolds with nonpositive curvature*, Topology **18** (1979), 361–380.

30. C. Sormani, *On loops representing elements of the fundamental group of a complete manifold with nonnegative Ricci curvature*, Indiana Univ. Math. J. **50** (2001), no. 4, 1867–1883.

31. S. Volpi, *Spectral properties of schrödinger operators.*, Master Thesis (2008), Università degli studi dell'Insubria, Como.

32. G. Wei and W. Wylie, *Comparison geometry for the smooth metric measure spaces*, Proceedings of the 4th International Congress of Chinese Mathematicians **II** (2007), 191–202.

33. ———, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405.

34. S. W. Wei, *The minima of the p-energy functional*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, 1996, pp. 171–203.

35. W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1803–1806.

36. N. Yang, *A note on nonnegative Bakry-Émery Ricci curvature*, Arch. Math. (Basel) **93** (2009), no. 5, 491–496.

Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, I-20133 Milano, ITALY

E-mail address: michele.rimoldi@gmail.com

Département de Mathématiques Université de Cergy–Pontoise, Site de Saint Martin 2, avenue Adolphe Chauvin 95302 Cergy–Pontoise Cedex France

E-mail address: giona.veronelli@gmail.com